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Rigidity theorems for closed hypersurfaces in a unit sphere ☆

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Abstract

In this paper, we prove some rigidity theorems for closed hypersurfaces in a unit sphere.

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1. Introduction

Let M^n be an n-dimensional closed hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$ and denote by S the squared norm of the second fundamental form of M^n. Many metric and topological rigidity theorems about M^n have been obtained. Simons [19], Chern

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et al. [10] and Lawson [11] proved that if M^n is minimal and if $S \leq n$ then M^n is either totally geodesic or a Clifford minimal hypersurface. Further results about rigidity of minimal hypersurfaces in $S^{n+1}(1)$ can be found, e.g., in [15–17], etc. As a natural generalization, the rigidity phenomenon for hypersurfaces in $S^{n+1}(1)$ with constant mean curvature has also been studied. It has been proven by Nomizu and Smyth [13] that if M^n has constant mean curvature and non-negative sectional curvature then M^n is either totally umbilic or a Riemannian product $S^k(c_1) \times S^{n-k}(c_2)$, $1 \leq k \leq n-1$, where $S^k(c)$ denotes the sphere of radius c. Alencar and do Carmo [2] have shown that if M^n has constant mean curvature H and if $S \leq nH^2 + C(H,n)$, where $C(H,n)$ is a constant that depends only on H and n, then M^n is either totally umbilic or a Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 \leq (n-1/n)$. On the other hand, some rigidity theorems for hypersurfaces with constant scalar curvature have been proven. It has been shown by Cheng and Yau that if M^n has non-negative sectional curvature and constant scalar curvature $n(n-1)r$ with $r \geq 1$, then M^n is isometric to either a totally umbilic hypersurface or a Riemannian product $S^k(c_1) \times S^{n-k}(c_2)$, $1 \leq k \leq n-1$ [9]. Li [12] proved that if M^n has constant scalar curvature $n(n-1)r$ with $r \geq 1$ and if $S \leq (n-1)(n(r-1)+2)/(n-2) + (n-2)/(n(r-1)+2)$, then M^n is isometric to either a totally umbilic hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 \leq (n-2)/(nr)$. Recently, Alencar et al. [3] obtained a gap theorem for closed hypersurfaces with constant scalar curvature $n(n-1)$ in a unit sphere.

In this paper, we prove a rigidity theorem for the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 \leq (n-1/n)$ without the constancy condition on the mean curvature or the scalar curvature. Namely, we have the following theorem.

Theorem 1.1. *Let M^n be an n-dimensional closed hypersurface in $S^{n+1}(1)$. Denote by S and H the squared norm of the second fundamental form and the mean curvature of M^n, respectively. Assume that the fundamental group $\pi_1(M^n)$ of M^n is infinite and that $S \leq S(n,H)$, where*

$$S(n,H) = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)|H|}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)}. \tag{1.1}$$

Then S is constant, $S = S(n,H)$ and M is isometric to a Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 \leq (n-1)/n$.

Montiel and Ros [14] have proven that an embedded closed hypersurface in a half-sphere with constant r-mean curvature for some $r \in \{1,\ldots,n\}$, is a totally umbilical hypersphere. In the second part of this paper, we give another characterization for the totally umbilical hypersphere.

Theorem 1.2. *Let M^n be an n-dimensional closed orientable hypersurface in $S^{n+1}(1)$. Assume that the squared norm S of the second fundamental form of M^n is constant and that one of the following conditions holds:*

(a) *The mean curvature H of M^n satisfies $H^2 \geq \{(n-1)/n^2\}S$ on M^n.*

(b) *The Ricci curvatures of M^n are bounded from below by $n-1$.*

Then M^n is a totally umbilical hypersphere.

2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold. We choose a local frame of orthonormal vector fields $\{e_1, \ldots, e_n\}$ on M^n and let $\{\omega_1, \ldots, \omega_n\}$ be the dual coframe. The connection forms $\{\omega_{ij}\}$ of M^n are characterized by the structure equations:

$$\mathrm{d}\omega_i = \sum_j \omega_{ij} \wedge \omega_j,\ \omega_{ij} + \omega_{ji} = 0, \tag{2.1}$$

$$\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.2}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

For any C^2-function f defined on M^n, we define its gradient and Hessian by the following formulas

$$\mathrm{d}f = \sum_i f_i\omega_i, \tag{2.3}$$

$$\sum_j f_{ij}\omega_j = \mathrm{d}f_i + \sum_j f_j\omega_{ji}. \tag{2.4}$$

The Laplacian of f is given by $\Delta f = \sum_i f_{ii}$. Let $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n. The covariant derivative of ϕ is defined by

$$\sum_k \phi_{ijk}\omega_k = \mathrm{d}\phi_{ij} + \sum_k \phi_{kj}\omega_{ki} + \sum_k \phi_{ik}\omega_{kj}. \tag{2.5}$$

Let $|\phi|^2 = \sum_{i,j} \phi_{ij}^2$ and $\mathrm{tr}\,\phi = \sum_i \phi_{ii}$. Choose a frame field $\{\omega_1, \ldots, \omega_n\}$ so that $\phi_{ij} = \lambda_i\delta_{ij}$. If ϕ satisfies the "Codazzi equation"

$$\phi_{ijk} = \phi_{ikj},$$

then we have [9]

$$\frac{1}{2}\Delta|\phi|^2 = \sum_{i,j,k} \phi_{ijk}^2 + \sum_i \lambda_i(\mathrm{tr}\,\phi)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.6}$$

The Weyl curvature tensor $W = (W_{ijkl})$ of M^n is defined by

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \end{aligned} \tag{2.7}$$

where R_{ij} and R are the components of Ricci curvature tensor and the scalar curvature of M^n, respectively. The Beck tensor $B = (B_{ijk})$ of M^n is given by

$$B_{ijk} = \frac{1}{n-2}(R_{ijk} - R_{ikj}) - \frac{1}{2(n-1)(n-2)}(\delta_{ij}R_k - \delta_{ik}R_j), \tag{2.8}$$

where R_{ijk} are the components of the covariant derivative of the Ricci curvature tensor of M^n and $R_k = e_k R$.

The following fact is needed in the proof of Theorem 1.1.

Lemma 2.1. [1] *If the Ricci curvature of a compact Riemannina manifold is non-negative and positive at a point, then the manifold carries a metric of positive Ricci curvature.*

M^n is said to be locally conformally flat if, for each $x \in M^n$, there exists a conformal diffeomorphism of a neighborhood of x onto an open set of the Euclidean n-space R^n. When $n \geq 4$, M^n is locally conformally flat if and only if $W_{ijkl} = 0$, and that $B_{ijkl} = 0$ on M^n in this case. When $n = 3$, we always have $W_{ijkl} = 0$ on M^3 and M^3 is a locally conformally flat Riemannian manifold if and only if $B_{ijk} = 0$. Thus, if M^n is a locally conformally flat Riemannian manifold, then

$$R_{ijkl} = \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) - \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.9}$$

and

$$R_{ijk} - \frac{1}{2(n-1)}\delta_{ij}R_k = R_{ikj} - \frac{1}{2(n-1)}\delta_{ik}R_j. \tag{2.10}$$

Let M^n be a hypersurface in a unit $(n+1)$-dimensional sphere $S^{n+1}(1)$ and denote by $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ the second fundamental form of M^n. The squared norm S of h and the mean curvature H of M^n are given by

$$S = \sum_{i,j} h_{ij}^2, \quad nH = \sum_i h_{ii},$$

respectively. The Gauss equation of M^n can be written as

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.11}$$

The Ricci tensor and the scalar curvature R of M^n are then given by

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.12}$$

$$R = n(n-1) + n^2H^2 - S, \tag{2.13}$$

respectively. The covariant derivative h_{ijk} of h satisfies $h_{ijk} = h_{ikj}$. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of (h_{ij}) are the principal curvatures of M^n.

3. Proofs of the results

Proof of Theorem 1.1. Assume that $\{e_1, \ldots, e_n\}$ diagonalizes the second fundamental form of M^n so that $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i \lambda_i \omega_i \otimes \omega_i$. In this case, we have

$$R_{ijkl} = (1 + \lambda_i \lambda_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{3.1}$$

$$R_{ij} = 0, i \neq j, \tag{3.2}$$

and

$$R_{ii} = n - 1 + nH\lambda_i - \lambda_i^2, i = 1, \ldots, n. \tag{3.3}$$

For any fixed $j \in \{1, \ldots, n\}$, since

$$(nH - \lambda_j)^2 = \left(\sum_{j \neq k} \lambda_k\right)^2 \leq (n-1)\sum_{k \neq j} \lambda_k^2 = (n-1)(S - \lambda_j^2),$$

we have

$$n^2H^2 - (n-1)S + n\lambda_j^2 - 2nH\lambda_j \leq 0. \tag{3.4}$$

It is easy to see from

$$\sum_i (\lambda_i - H) = 0, \sum_i (\lambda_i - H)^2 = S - nH^2,$$

that

$$(\lambda_j - H)^2 \leq \frac{n-1}{n}(S - nH^2),$$

which, combining with (3.4), implies that

$$\begin{aligned} 0 &\geq n(\lambda_j^2 - nH\lambda_j) + (n-2)n(\lambda_j - H)H + 2(n-1)nH^2 - (n-1)S \\ &\geq n(\lambda_j^2 - nH\lambda_j) - (n-2)n|H|\sqrt{\frac{n-1}{n}(S - nH^2)} + 2(n-1)nH^2 - (n-1)S. \end{aligned} \tag{3.5}$$

It then follows from (3.3) that

$$\begin{aligned} R_{jj} &\geq (n-1) - (n-2)|H|\sqrt{\frac{n-1}{n}(S - nH^2)} + 2(n-1)H^2 - \frac{(n-1)}{n}S \\ &= \frac{n-1}{n}\left(n + 2nH^2 - S - (n-2)|H| \cdot \sqrt{\frac{n}{n-1}} \cdot \sqrt{S - nH^2}\right) \\ &= \frac{n-1}{n}\left(\sqrt{S - nH^2} + \frac{1}{2}\sqrt{\frac{n}{n-1}}\left((n-2)|H| + \sqrt{n^2H^2 + 4(n-1)}\right)\right) \end{aligned}$$

$$\times \left(-\sqrt{S-nH^2}+\frac{1}{2}\sqrt{\frac{n}{n-1}}\left(-(n-2)|H|+\sqrt{n^2H^2+4(n-1)}\right)\right). \tag{3.6}$$

Observe that our condition $S \leq S(n, H)$ is equivalent to

$$S-nH^2 \leq \left(\frac{1}{2}\sqrt{\frac{n}{n-1}}\left(-(n-2)|H|+\sqrt{n^2H^2+4(n-1)}\right)\right)^2,$$

that is

$$-\sqrt{S-nH^2}+\frac{1}{2}\sqrt{\frac{n}{n-1}}\left(-(n-2)|H|+\sqrt{n^2H^2+4(n-1)}\right) \geq 0.$$

Thus

$$R_{jj} \geq 0, \forall j,$$

which, combining with (3.2), implies that M^n has non-negative Ricci curvature. Since our M^n has infinite fundamental group, we conclude from Bonnet–Myer's theorem [7] and Lemma 2.1 that $\forall p \in M^n$, there exists a unit vector $u \in T_pM^n$, such that the Ricci curvature Ric of M^n satisfies $\mathrm{Ric}(u, u) = 0$. Note that Ric attains its maximum and minimum in the principal directions. We can assume without loss of generality that at any fixed point $x \in M^n$, $R_{nn} = 0$. Thus, when $n = j$, at the point x, the inequality (3.5) should be an equality and $S(x) = S(n, H)(x)$, which in turn implies that when $n = j$, the above inequalities should take equality sign at x. Consequently, we know that $\lambda_1(x) = \cdots = \lambda_{n-1}(x)$. Since $x \in M^n$ is arbitrary, one then deduces that our M^n has at most two distinct principal curvatures and, when M^n has exactly two distinct principal curvatures, one of these two distinct principal curvatures should be simple. Let us assume that $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$ and $\lambda_n = \mu$ on M^n. Since $R_{ii} \geq 0$ and one of $R_{11}, \ldots, R_{nn}$ is zero, we deduce from (3.3) that $1 + \lambda\mu = 0$. Therefore, we conclude that M^n has exactly two distinct principal curvatures one of which is simple and $S = S(n, H)$ holds on M. We want to show that λ and μ are constant functions on M. Notice that our M^n is a closed manifold with non-negative Ricci curvature. It is well-known that [18, p. 220] the Riemannian universal covering space $\tilde{M}^n$ of M^n can be decomposed as $\bar{M}^{n-s} \times R^s$ for some $s \in \{0, 1, \ldots, n\}$, where $\bar{M}^{n-s}$ is a closed simply connected $(n - s)$-dimensional Riemannina manifold with non-negative Ricci curvature and R^s is the s-dimensional Euclidean space with its standard flat metric. The infinity of $\pi_1(M^n)$ implies that $\bar{M}^{n-s} \times R^s$ is non-compact and so we have $s \geq 1$. Since $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$, $\lambda_n = \mu$ and $1 + \lambda\mu = 0$ on M^n, we know from (3.1) that if $u = \sum_{i=1}^n a_ie_i$, $v = \sum_{j=1}^n b_je_j \in T_pM^n$ with $|u| = |v| = 1$, $\langle u, v\rangle = 0$, then the sectional curvature $K(u \wedge v)$ of the plane spanned by u and v is given by

$$K(u \wedge v) = \sum_{i,j,k,l} a_i b_j a_k b_l R_{ijkl} = \sum_{i,j,k,l} a_i b_j a_k b_l (1 + \lambda_i \lambda_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$
$$= 1 + \left(\sum_i \lambda_i a_i^2\right)\left(\sum_j \lambda_j b_j^2\right) - \left(\sum_i \lambda_i a_i b_i\right)^2$$
$$= 1 + \left(\lambda(1 - a_n^2) + \mu a_n^2\right)\left(\lambda(1 - b_n^2) + \mu b_n^2\right) - (\lambda(-a_n b_n) + \mu a_n b_n)^2$$
$$= (1 + \lambda^2)(1 - a_n^2 - b_n^2). \quad \square \tag{3.7}$$

Lemma 3.1. *A vector $u \in T_pM^n$ with $|u| = 1$ satisfies the following condition*

$$(*) \text{ if } v \in T_pM^n \text{ with } \langle u, v\rangle = 0, |v| = 1, \text{ then } K(u \wedge v) = 0,$$

if and only if $u = \pm e_n(p)$.

Proof of Lemma 3.1. If $u = \pm e_n(p)$ and v satisfies $|v| = 1$, $\langle u, v\rangle = 0$, then we can write

$$v = \sum_{i=1}^{n-1} a_i e_i(p), \sum_{i=1}^{n-1} a_i^2 = 1.$$

Thus

$$K(u \wedge v) = \sum_{i,j} a_i a_j R_{ninj} = \sum_{i,j} a_i a_j (1 + \lambda_n \lambda_i)(\delta_{nn}\delta_{ij} - \delta_{ni}\delta_{nj})$$
$$= \sum_i (1 + \mu\lambda_i) a_i^2 - a_n^2(1 + \mu^2) = 0. \tag{3.8}$$

On the other hand, if $(*)$ is satisfied and suppose by contradiction that

$$u = a e_n(p) + w, \tag{3.9}$$

where $\langle w, e_n(p)\rangle = 0$, $w \neq 0$. Let $w = \sum_{i=1}^{n-1} c_i e_i(p)$ and take a vector $z = \sum_{i=1}^{n-1} d_i e_i(p) \in T_pM^n$ satisfying $|z| = 1$, $\langle z, w\rangle = 0$. Then $\langle z, u\rangle = 0$ and so we have from (3.7) that

$$K(u \wedge z) = (1 + \lambda^2)(1 - a^2) = (1 + \lambda^2)|w|^2 > 0,$$

which contradicts to $(*)$. Thus u is parallel to $e_n(p)$, since $|u| = 1$, we know that $u = \pm e_n(p)$. Lemma 3.1 is proved. $\square$

Let us go on the proof of Theorem 1.1. Since M and $\bar{M}^{n-s} \times R^s$ are locally isometric, it follows from Lemma 3.1 that $s = 1$. We claim that $\bar{M}^{n-1}$ has constant sectional curvature and so is isometric to a Euclidean $(n-1)$-sphere. In order to see this, let us assume first that $n \geq 4$. Let $\pi : \bar{M}^{n-1} \times R \to M$ be the natural projection; then π is a local isometry. For any $x \in \bar{M}^{n-1}$, take an orthonormal base $\{f_1, \ldots, f_{n-1}\}$ of $T_x\bar{M}^{n-1}$. Since $T_{(x,0)}(\bar{M}^{n-1} \times R) = T_x\bar{M}^{n-1} \times T_0R$, we know that $\{(f_1, 0), \ldots, (f_{n-1}, 0), (\mathbf{0}, 1)\}$

is an orthonormal base of $T_{(x,0)}(\bar{M}^{n-1} \times R)$, where $\mathbf{0}$ is the zero-vector of $T_x\bar{M}^{n-1}$. Observe that for any $v \in T_{(x,0)}(\bar{M}^{n-1} \times R)$ with $\langle v, (\mathbf{0}, 1)\rangle = 0$, $|v| = 1$, it holds $\tilde{K}((\mathbf{0}, 1) \wedge v) = 0$, where $\tilde{K}$ denotes the sectional curvature of $\bar{M}^{n-1} \times R$. It follows that $K(\mathrm{d}\pi_{(x,0)}((\mathbf{0}, 1)) \wedge z) = 0$, $\forall z \in T_{\pi(x,0)}M^n$ with $|z| = 1$ and $\langle \mathrm{d}\pi_{(x,0)}((\mathbf{0}, 1)), z\rangle = 0$. Thus we know from Lemma 3.1 that $\mathrm{d}\pi_{(x,0)}((\mathbf{0}, 1)) = \pm e_n(y)$ and so for any $j = 1, \ldots, n-1$, $\mathrm{d}\pi_{(x,0)}((f_j, 0)) \in \mathrm{span}\{e_1(y), \ldots, e_{n-1}(y)\}$, where $y = \pi(x, 0)$. Hence, $\tilde{K}((f_i, 0) \wedge (f_j, 0)) = K(\mathrm{d}\pi_{(x,0)}((f_i, 0)) \wedge \mathrm{d}\pi_{(x,0)}((f_j, 0))) = 1 + \lambda^2(y)$, $i \neq j \in \{1, \ldots, n-1\}$, which shows that for any $x \in \bar{M}^{n-1}$ and any two-dimensional plane $P \subset T_x\bar{M}^{n-1}$, the sectional curvature $\bar{K}(P)$ of $\bar{M}^{n-1}$ on P must satisfy $\bar{K}(P) = \tilde{K}(P) = g(x) > 0$, where $g(x) = 1 + \lambda^2(\pi(x, 0))$ is a function on $\bar{M}^{n-1}$. Thus, since $\dim(\bar{M}^{n-1}) \geq 3$, we know from the well-known Schur Lemma [4, p. 106] that $\bar{M}^{n-1}$ has constant sectional curvature.

Consider now the case that $n = 3$. Let us first show that (2.10) is satisfied and so M^3 is a locally conformally flat manifold. In fact, since $n = 3$, we get by taking the covariant derivatives of (2.12) that

$$R_{ijk} = 3H_k h_{ij} + 3H h_{ijk} - \sum_l (h_{ilk}h_{lj} + h_{il}h_{ljk}). \tag{3.10}$$

Also, one has from (2.13) that

$$R_k = 18HH_k - S_k. \tag{3.11}$$

Hence

$$\begin{aligned} B_{ijk} &= R_{ijk} - R_{ikj} - \frac{1}{4}(\delta_{ij}R_k - \delta_{ik}R_j) \\ &= 3H_k h_{ij} - 3H_j h_{ik} + \sum_l (h_{ilj}h_{lk} - h_{ilk}h_{lj}) \\ &\quad - \frac{1}{4}\left(\delta_{ij}(18HH_k - S_k) - \delta_{ik}(18HH_j - S_j)\right). \end{aligned}$$

From $h_{ij} = \lambda_i\delta_{ij}$, we have

$$S_k = 2\sum_{l,m} h_{lm}h_{lmk} = 2\sum_l \lambda_l h_{llk}.$$

Thus

$$\begin{aligned} B_{ijk} &= 3H_k\left(\lambda_i - \frac{3}{2}H\right)\delta_{ij} - 3H_j\left(\lambda_i - \frac{3}{2}H\right)\delta_{ik} + h_{ikj}(\lambda_k - \lambda_j) + \frac{1}{4}(\delta_{ij}S_k - \delta_{ik}S_j) \\ &= 3(H_k\delta_{ij} - H_j\delta_{ik})\left(\lambda_i - \frac{3}{2}H\right) + h_{ikj}(\lambda_k - \lambda_j) + \frac{1}{2}\sum_l (\delta_{ij}h_{llk} - \delta_{ik}h_{llj})\lambda_l. \end{aligned} \tag{3.12}$$

Taking $h_{lm} = \lambda_l \delta_{lm}$ in the equality

$$\mathrm{d}h_{ij} + \sum_l (h_{lj}\omega_{li} + h_{il}\omega_{lj}) = \sum_l h_{ijl}\omega_l, \tag{3.13}$$

one easily gets

$$h_{ijk} = \delta_{ij} e_k \lambda_i + (\lambda_i - \lambda_j)\omega_{ij}(e_k), \tag{3.14}$$

and so we have

$$h_{iik} = e_k \lambda_i. \tag{3.15}$$

Since $h_{ijk} = h_{ikj}$, we know from (3.14) that if i, j, k are all distinct, then

$$(\lambda_i - \lambda_j)\omega_{ij}(e_k) = (\lambda_i - \lambda_k)\omega_{ik}(e_j). \tag{3.16}$$

Since $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \mu$, we conclude that when i, j, k are all distinct

$$B_{ijk} = (\lambda_k - \lambda_j)h_{ikj} = (\lambda_k - \lambda_j)(\lambda_i - \lambda_k)\omega_{ik}(e_j) = (\lambda_k - \lambda_j)(\lambda_i - \lambda_j)\omega_{ij}(e_k) = 0. \tag{3.17}$$

It is trivial to see from (3.12) that $B_{iii} = 0$. Let A be the Weingarten operator defined by the second fundamental form, that is, for any $p \in M$ and all $X, Y \in T_pM$, $A : T_pM \to T_pM$, $\langle AX, Y\rangle = h(X, Y)$. Set

$$\mathcal{D}_p(\lambda) = \{X \in T_pM : AX = \lambda_p X\}$$

and let $\mathcal{D}(\lambda)$ be the assignment of $\mathcal{D}_p(\lambda)$ to each point $p \in M$. Since the multiplicity of the principal curvature λ is greater than one, it follows from [15] that $\mathcal{D}(\lambda)$ is a completely integrable distribution on M and that λ is constant on each leaf of $\mathcal{D}(\lambda)$. Thus we have $e_1\lambda = e_2\lambda = 0$. If $i \neq k$, we have from (3.12) and (3.15) that

$$\begin{aligned} B_{iik} &= 3H_k\left(\lambda_i - \frac{3}{2}H\right) + h_{iik}(\lambda_k - \lambda_i) + \frac{1}{2}\sum_l h_{llk}\lambda_l \\ &= (2e_k\lambda + e_k\mu)\left(\lambda_i - \lambda - \frac{1}{2}\mu\right) + (e_k\lambda_i)(\lambda_k - \lambda_i) + \frac{1}{2}\sum_l h_{llk}\lambda_l. \end{aligned} \tag{3.18}$$

Thus

$$B_{113} = B_{223} = (2e_3\lambda + e_3\mu)\left(-\frac{1}{2}\mu\right) + (e_3\lambda)(\mu - \lambda) + (e_3\lambda)\lambda + \frac{1}{2}(e_3\mu)\mu = 0,$$

$$B_{331} = e_1\mu\left(\frac{\mu}{2} - \lambda\right) + (e_1\mu)(\lambda - \mu) + \frac{1}{2}(e_1\mu)\mu = 0.$$

Similarly, we have

$$B_{332} = 0,\ B_{ikk} = B_{iki} = 0,\ i \neq k.$$

Therefore, M^3 is locally conformally flat and so $\tilde{M}^3$ is also locally conformally flat. Recall that $\tilde{M}^3 = \bar{M}^2 \times R$, where $\bar{M}^2$ has non-negative Gaussian curvature κ. We shall use the same notations R_{ijkl} and R_{ij}, etc. to denote the components of the curvature tensor and the Ricci curvature tensor, etc. of $\tilde{M}^3$, respectively. Take an orthonormal local frame $\{v_1, v_2, v_3\}$ of $\tilde{M}^3$ such that v_1 and v_2 are tangent to $\bar{M}^2$. Since $\tilde{M}^3$ is a product, one can see easily that

$$R_{11} = R_{22} = R_{1212} + R_{1313} = R_{1212} = \kappa,$$

$$R_{12} = R_{13} = R_{23} = R_{33} = 0, \quad R = 2\kappa.$$

Taking $i = j = 1$, $k = 2$ in the equality (2.10), we get

$$R_{112} - R_{121} = \frac{1}{4} R_2.$$

From the definition, we have

$$R_{121} = (\mathrm{d}R_{12})(v_1) + \sum_{l=1}^{3} R_{l2}\omega_{l1}(v_1) + \sum_{l=1}^{3} R_{1l}\omega_{l2}(v_1) = 0,$$

$$R_{112} = (\mathrm{d}R_{11})(v_2) + \sum_{l=1}^{3} R_{l1}\omega_{l1}(v_2) + \sum_{l=1}^{3} R_{l2}\omega_{l2}(v_2) = v_2 R_{11} = v_2\kappa.$$

Therefore,

$$v_2\kappa = \frac{1}{2} v_2\kappa,$$

and so

$$v_2\kappa = 0.$$

Similarly, we have $v_1\kappa = 0$. Thus κ is a constant function. Hence, for any $n \geq 3$, $\bar{M}^{n-1}$ is a Euclidean sphere. Consequently our M^n has constant scalar curvature. But the scalar curvature of M^n is given by

$$r = \sum_i R_{ii} = (n-1)(n-2)(1+\lambda^2).$$

Hence λ is a constant and so $\mu = -1/\lambda$ is also constant. That is, M^n is an isoparametric hypersurface in $S^{n+1}(1)$ with two distinct principal curvatures one of which is simple. From the work of Cartan [6], we conclude that $M^n = S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ for some $c \in (0, 1)$. Since the principal curvatures of $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ are $\lambda_1 = \cdots = \lambda_{n-1} =$

$\sqrt{1/(c^2-1)}$, $\lambda_n = -\sqrt{c^2/(1-c^2)}$ and $S = S(n, H)$, we know that $c^2 \le (n-1/n)$. This completes the proof of Theorem 1.1.

Before proving Theorem 1.2, we recall an algebraic fact.

Lemma 3.2. [8] *Let $x_1, \dots, x_n$ be $n(\ge)2$ real numbers satisfying the inequality $(\sum_i x_i)^2 \ge (n-1)\sum_i x_i^2$. Then for any distinct i, j, $1 \le i < j \le n$, we have $x_i x_j \ge 0$.*

Proof of Theorem 1.2. Let $h = \sum_{i,j=1}^n h_{ij}\omega_i \otimes \omega_j$ be the second fundamental form of M^n and assume that $h_{ij} = \lambda_i \delta_{ij}$, where λ_i, $i = 1, \dots, n$ are the principal curvatures of M^n. It follows from the Gauss equation that for any $i \neq j$

$$R_{ijij} = 1 + \lambda_i \lambda_j, \tag{3.19}$$

Let $|h|^2 = \sum_{i,j=1}^n h_{ij}^2 = \sum_{i=1} \lambda_i^2$ and $\mathrm{tr}\, h = \sum_{i=1}^n \lambda_i$. Since $|h|^2$ is constant, we have from (2.6) that

$$0 = \sum_{i,j,k=1}^n h_{ijk}^2 + \sum_{i=1}^n \lambda_i (\mathrm{tr}\, h)_{ii} + \frac{1}{2}\sum_{i,j=1}^n R_{ijij}(\lambda_i - \lambda_j)^2. \tag{3.20}$$

Consider first the case that $H^2 \ge \{(n-1)/n^2\}S$. In this case, we know from Lemma 2.1 that $\lambda_i \lambda_j \ge 0$, $\forall i \neq j$. Thus one finds from (3.19) that the sectional curvatures of M^n are bounded from below by 1. It then follows from Theorem 1.1 in [5] that either M^n is totally geodesic or M^n is the boundary of a convex body in an open half-sphere, which implies that the second fundamental form of M^n is always semi-positive definite if we choose the unit normal vector field of M^n properly. Therefore, we can assume that $\lambda_i \ge 0$ on M^n, $\forall i = 1, \dots, n$.

Take a point $p \in M^n$ such that

$$(\mathrm{tr}\, h)(p) = \min_{x \in M^n} (\mathrm{tr}\, h)(x).$$

Then we have from the maximal principle that

$$(\mathrm{tr} h)_{ii}(p) \ge 0, i = 1, \dots, n, \tag{3.21}$$

which, combining with (3.20), gives

$$0 \ge \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2(p),$$

and so

$$\lambda_1(p) = \cdots = \lambda_n(p). \tag{3.22}$$

Now for any $q \in M^n$, we have from $S(q) = S(p)$ and (3.22) that

$$\sum_{i,j}(\lambda_i(q)-\lambda_j(q))^2 = 2n\sum_i(\lambda_i(q))^2 - 2\left(\sum_i^n \lambda_i(q)\right)^2 \le 2n\sum_i^n(\lambda_i(p))^2 - 2\left(\sum_i^n \lambda_i(p)\right)^2 = 0. \quad (3.23)$$

Thus M^n is totally umbilical. Assume now that the Ricci curvature of M^n is bounded from below by $n-1$. Then we have from Gauss equation that

$$\left(\sum_{i=1}^n \lambda_i\right)\lambda_j - \lambda_j^2 \ge 0,\ j=1,\dots,n, \text{ on } M^n. \quad (3.24)$$

Suppose without loss of generality that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. We claim that

$$\lambda_i\lambda_j \ge 0, \forall 1 \le i, j \le n, \text{ on } M^n, \quad (3.25)$$

which, combining with the above arguments above, will imply that M^n is totally umbilical. Let us verify that for any $p \in M^n$, there holds either

$$\lambda_i(p) \ge 0, i = 1,\dots,n,$$

or

$$\lambda_i(p) \le 0, i = 1,\dots,n,$$

and so (3.25) holds. We shall prove this fact by contradiction. Thus suppose that there exists a $q \in M^n$ such that

$$\lambda_1(q) > 0 \quad \text{and} \quad \lambda_n(q) < 0.$$

If $\sum_{i=1}^n \lambda_i(q) \ge 0$, then we have

$$\sum_{i=1}^{n-1}\lambda_i(q) \ge -\lambda_n(q) > 0.$$

Thus

$$\left(\sum_{i=1}^n \lambda_i(q)\right)\lambda_n(q) - \lambda_n^2(q) = \left(\sum_{i=1}^{n-1}\lambda_i(q)\right)\lambda_n(q) < 0,$$

which contradicts to (3.24). On the other hand, if $\sum_{i=1}^{n} \lambda_i(q) < 0$, then

$$\sum_{i=2}^{n} \lambda_i(q) < -\lambda_1(q) < 0,$$

which gives

$$\left(\sum_{i=1}^{n} \lambda_i(q)\right) \lambda_1(q) - \lambda_1^2(q) = \left(\sum_{i=2}^{n} \lambda_i(q)\right) \lambda_1(q) < 0,$$

contradicting to (3.24) again. This completes the proof of Theorem 1.2. □

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